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LETTER TO THE EDITOR

On the principal subalgebra of quantum enveloping algebras $gl_q(l+1)$

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Abstract. The existence of a principal subalgebra of type $sl_q(2)$ for quantum enveloping algebras $gl_q(l+1)$ or $sl_q(l+1)$ is investigated. Surprisingly, only when $l=2$ and when all relations are restricted to symmetric representations such a principal subalgebra happens to exist. This case, $sl_q(3)\supset sl_q(2)$, is the q-deformation of the classical $su(3)\supset so(3)$ embedding for symmetric su(3) representations, and is analysed in more detail, giving a connection with q-deformed spherical harmonics.

Principal three-dimensional subalgebras for simple Lie algebras were introduced by Dynkin (1957) and Kostant (1959). They have many important applications in mathematics, being related to the exponents of simple Lie groups (Kostant 1959) and to various combinatorial results obtained by Hughes (1977) and later generalized by Stanley (1980). In various physical models, the principal three-dimensional subalgebra plays a crucial role, since it is usually the subalgebra describing the angular momentum of the system (Hammermesh 1962). As examples, we mention here: Elliott's model $SU(3)\supset SO(3)$ (Elliott 1958); quadrupole vibrations of the nucleus (Bohr 1952, Chacón *et al* 1976) or octupole vibrations in which the chains $U(5)\supset O(5)\supset O(3)$ or $U(7)\supset O(7)\supset G_2\supset O(3)$ appear (these appear also in atomic spectroscopy (Judd 1963)); the interacting boson model (Arima and Iachello 1976) has dynamical symmetries in which $U(3)\supset O(3)$ and $U(5)\supset O(5)\supset O(3)$ appear.

Let G_l be a simple Lie algebra of rank l, with Chevalley generators $\{e_i, f_i, h_i|i=1,2,\ldots,l\}$. A principal three-dimensional subalgebra of G_l is a subalgebra A of type sl(2), with basis $\{E, F, H\}$ satisfying

$$[H,E]=2E \qquad [H,F]=-2F \qquad [E,F]=H \tag{1}$$

such that the number of irreducible components occurring in the complete reduction of the adjoint representation of G_l with respect to A is equal to l (Kostant 1959). The Lie algebra G_l has an involutive antiautomorphism σ defined by $\sigma(h_i)=h_i$, $\sigma(e_i)=f_i$ and $\sigma(f_i)=e_i$, which is related to Hermitian conjugation; if the principal subalgebra A is required to be invariant under σ, i.e. $\sigma(H)=H$, $\sigma(E)=F$ and $\sigma(F)=E$, then the elements of A have a unique expression in terms of the generators $\{e_i, f_i, h_i\}$. For

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$G_l = A_l = \mathrm{sl}(l+1)$, one obtains

$$H = \sum_{i=1}^{l} i(l+1-i)h_i$$
$$E = \sum_{i=1}^{l} \sqrt{i(l+1-i)}\, e_i \tag{2}$$
$$F = \sum_{i=1}^{l} \sqrt{i(l+1-i)}\, f_i.$$

For $\mathrm{gl}(l+1)$, the Cartan subalgebra contains $l+1$ basis elements $N_0, N_1, \ldots, N_l$, which are related to the l basis elements h_i of $\mathrm{sl}(l+1)$ by $h_i = N_{i-1} - N_i$ $(i = 1, 2, \ldots, l)$. Thus the principal subalgebra of $\mathrm{gl}(l+1)$ has the same form as (2), except that the diagonal element becomes

$$H = \sum_{i=0}^{l} (l-2i) N_i. \tag{3}$$

Quantum enveloping algebras are certain q-deformations of enveloping algebras of simple Lie algebras, being at the centre of much attention recently (e.g. Doebner and Hennig 1990). So far, however, very little work has been done in studying non-trivial subalgebras of quantum enveloping algebras (see e.g. Dobrev 1990). In this letter the investigation of principal subalgebras of quantum enveloping algebras of type $\mathrm{gl}_q(l+1)$ or $\mathrm{sl}_q(l+1)$ is initiated. The algebra $\mathrm{gl}_q(l+1)$ is the associative algebra spanned by generators e_i, f_i $(i = 1, 2, \ldots, l)$ and N_i $(i = 0, 1, \ldots, l)$ subject to the relations (Jimbo 1986)

$$\begin{aligned}
&[N_i, N_j] = 0\\
&[N_i, e_j] = (\delta_{i,j-1} - \delta_{ij}) e_j \qquad [N_i, f_j] = -(\delta_{i,j-1} - \delta_{ij}) f_j\\
&[e_i, f_j] = \delta_{ij}[N_{i-1} - N]\\
&\sum_{k=0}^{1+\delta_{i-1,j}+\delta_{i,j-1}} (-1)^k \begin{bmatrix} 1+\delta_{i-1,j}+\delta_{i,j-1} \\ k \end{bmatrix} e_i^{1+\delta_{i-1,j}+\delta_{i,j-1}-k} e_j e_i^k = 0 \qquad (i \neq j)\\
&\sum_{k=0}^{1+\delta_{i-1,j}+\delta_{i,j-1}} (-1)^k \begin{bmatrix} 1+\delta_{i-1,j}+\delta_{i,j-1} \\ k \end{bmatrix} f_i^{1+\delta_{i-1,j}+\delta_{i,j-1}-k} f_j f_i^k = 0 \qquad (i \neq j)
\end{aligned} \tag{4}$$

where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{[x]!}{[y]![x-y]!}$$

and $[x]! = [x][x-1]\ldots[1]$. In the limit $q \to 1$, this reduces to the universal enveloping algebra of $\mathrm{gl}_q(l+1)$.

A principal subalgebra of $\mathrm{gl}_q(l+1)$ is defined as follows: it is a subalgebra of $\mathrm{gl}_q(l+1)$ of type $\mathrm{sl}_q(2)$, i.e. its generators $\{E, F, H\}$ satisfy

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = [H] \tag{5}$$

and in the limit $q \to 1$, this $\mathrm{sl}_q(2)$ subalgebra reduces to the principal subalgebra of $\mathrm{gl}(l+1)$.

In this letter it is proved that principal subalgebras of $\mathrm{gl}_q(l+1)$ do not exist in general. It is shown, however, that when all relations are restricted to the totally symmetric representations of $\mathrm{gl}_q(l+1)$, a principal subalgebra does exist in the case $l=2$ but not for other l-values. The non-triviality of the solution with $l=2$ can be seen from the curious q-factors arising in (21). Some further aspects of $\mathrm{gl}_q(3)\supset \mathrm{sl}_q(2)$, which is the q-deformation of $\mathrm{su}(3)\supset \mathrm{so}(3)$, are discussed. A remarkable relation between q-numbers, equation (20), is obtained as a byproduct.

Since H is an ordinary diagonal operator in (5), and because of the limiting case (2) or (3), it follows that the most general form of a principal subalgebra is:

$$H=\sum_{i=0}^{l}(l-2i)N_i \qquad E=\sum_{i=1}^{l}\mu_i(N_0,\ldots,N_l)e_i$$
$$F=\sum_{i=1}^{l}f_i\mu_i(N_0,\ldots,N_l) \tag{6}$$

where μ_i are (q-dependent) functions of $N_0,\ldots,N_l$, and invariance under σ has been assumed. The first two relations in (5) are satisfied by (6); the crucial relation to be satisfied is the third relation of (5). Let us concentrate for a moment on the case $l=2$, and rewrite (6) in the form

$$\begin{aligned}H&=2N_0-2N_2\\E&=\alpha(N_0N_1N_2)e_1+e_2\beta(N_0N_1N_2)\\F&=f_1\alpha(N_0N_1N_2)+\beta(N_0N_1N_2)f_2.\end{aligned} \tag{7}$$

When calculating $[E,F]$, the terms in e_1f_2 and e_2f_1 must vanish, leading to the following condition on the functions α and β:

$$\alpha(N_0N_1N_2)\beta(N_0-1,N_1+1,N_2)=\alpha(N_0,N_1+1,N_2-1)\beta(N_0N_1N_2). \tag{8}$$

Then, there comes

$$\begin{aligned}[E,F]=\alpha^2(N_0N_1N_2)e_1f_1-\alpha^2(N_0+1,N_1-1,N_2)f_1e_1\\+\beta^2(N_0,N_1-1,N_2+1)e_2f_2-\beta^2(N_0N_1N_2)f_2e_2.\end{aligned} \tag{9}$$

The right-hand side of (9) should be expressible in terms of N_0, N_1 and N_2 only. It follows from (4) that this is the case only when the coefficients of e_if_i and f_ie_i are equal, i.e.

$$\begin{aligned}\alpha(N_0+1,N_1-1,N_2)&=\alpha(N_0N_1N_2)\\\beta(N_0,N_1-1,N_2+1)&=\beta(N_0N_1N_2).\end{aligned} \tag{10}$$

Then (9) becomes

$$[E,F]=\alpha^2(N_0N_1N_2)[N_0-N_1]+\beta^2(N_0N_1N_2)[N_1-N_2] \tag{11}$$

such that the condition $[E,F]=[H]$ reduces to

$$\alpha^2(n_0n_1n_2)[n_0-n_1]+\beta^2(n_0n_1n_2)[n_1-n_2]=[2n_0-2n_2] \tag{12}$$

where we have substituted ordinary variables n_i for the operators N_i, which is allowed since the N_i are commuting operators. Replacing in (12) n_0 by n_0+1 and n_1 by n_1-1, and using (10), leads to

$$\alpha^2(n_0n_1n_2)[n_0-n_1+2]+\beta^2(n_0+1,n_1-1,n_2)[n_1-n_2-1]=[2n_0-2n_2+2]$$

or by iteration

$$\alpha^2(n_0n_1n_2)[n_0-n_1+2k]+\beta^2(n_0+k,\, n_1-k,\, n_2)[n_1-n_2-k]=[2n_0-2n_2+2k]. \tag{13}$$

Putting $k=n_1-n_2$ leads to

$$\alpha^2(n_0n_1n_2)=q^{n_0+n_1-2n_2}+q^{-n_0-n_1+2n_2} \tag{14}$$

which satisfies indeed the first equation of (10). Using this in (12) then implies that

$$\beta^2(n_0n_1n_2)=q^{n_1-n_2}+q^{-n_1\times n_2}. \tag{15}$$

In conjunction with the second equation of (10), this leads to a contradiction unless $q=1$. Thus we have shown that a general principal subalgebra does not exist for $\mathrm{gl}_q(3)$. For $l>2$, the conclusion is the same, but we shall not present a detailed treatment here.

Although a general principal subalgebra does not exist for $\mathrm{gl}_q(3)$, we shall show in this section that it does exist when all relations are restricted to the completely symmetric representations of $\mathrm{gl}_q(3)$. These representations are labelled by $\{N, 0, 0\}$ ($N\in\{0, 1, 2, \dots\}$), and the basis vectors are of the form $|n_0n_1n_2\rangle$ with $n_0+n_1+n_2=N$. The action of the $\mathrm{gl}_q(3)$ generators is given by (Jimbo 1986):

$$\begin{aligned}
&N_i|n_0n_1n_2\rangle=n_i|n_0n_1n_2\rangle \qquad (i=+1, 0, -1)\\
&e_1|n_0n_1n_2\rangle=\sqrt{[n_0+1][n_1]}\,|n_0+1,\, n_1-1,\, n_2\rangle\\
&e_2|n_0n_1n_2\rangle=\sqrt{[n_1+1][n_2]}\,|n_0,\, n_1+1,\, n_2-1\rangle\\
&f_1|n_0n_1n_2\rangle=\sqrt{[n_0][n_1+1]}\,|n_0-1,\, n_1+1,\, n_2\rangle\\
&f_2|n_0n_1n_2\rangle=\sqrt{[n_1][n_2+1]}\,|n_0,\, n_1-1,\, n_2+1\rangle.
\end{aligned} \tag{16}$$

The starting point is again (7), and the purpose is to find functions α and β such that (5) is satisfied on basis vectors of the form $|n_0n_1n_2\rangle$. The condition $[E, F]=[H]$ leads to the following two equations in α and β:

$$\alpha(n_0n_1n_2)\beta(n_0-1,\, n_1+1,\, n_2)=\alpha(n_0,\, n_1+1,\, n_2-1)\beta(n_0n_1n_2) \tag{17}$$

$$\begin{aligned}
&\alpha^2(n_0n_1n_2)[n_0][n_1+1]-\alpha^2(n_0+1,\, n_1-1,\, n_2)[n_0+1][n_1]+\beta^2(n_0,\, n_1-1,\, n_2+1)[n_1]\\
&\qquad\times[n_2+1]-\beta^2(n_0n_1n_2)[n_1+1][n_2]=[2n_0-2n_2].
\end{aligned} \tag{18}$$

It is remarkable that this set of equations in two unknown functions α and β has a solution, namely

$$\begin{aligned}
&\alpha(n_0n_1n_2)=q^{n_2-\frac{1}{2}n_1}\sqrt{q^{n_0}+q^{-n_0}}\\
&\beta(n_0n_1n_2)=q^{n_0-\frac{1}{2}n_1}\sqrt{q^{n_2}+q^{-n_2}}.
\end{aligned} \tag{19}$$

The verification of (18) depends upon a rather intriguing identity for q-numbers:

$$\begin{aligned}
&q^{2z-y}[2x][y+1]-q^{2z-y+1}[2x+2][y]+q^{2x-y+1}[y][2z+2]\\
&\qquad -q^{2x-y}[y+1][2z]=[2x-2z].
\end{aligned} \tag{20}$$

Thus we have shown that $\mathrm{gl}_q(3)$ contains three elements

$$\begin{aligned}
&H=2N_0-2N_2\\
&E=q^{N_2-\frac{1}{2}N_1}\sqrt{q^{N_0}+q^{-N_0}}e_1+e_2q^{N_0-\frac{1}{2}N_1}\sqrt{q^{N_2}+q^{-N_2}}\\
&F=f_1q^{N_2-\frac{1}{2}N_1}\sqrt{q^{N_0}+q^{-N_0}}+q^{N_0-\frac{1}{2}N_1}\sqrt{q^{N_2}+q^{-N_2}}f_2
\end{aligned} \tag{21}$$

which satisfy the relations (5) of an $\mathrm{sl}_q(2)$ algebra when acting upon symmetric representations of $\mathrm{gl}_q(3)$, and which tend to the principal sl(2) subalgebra of gl(3) in the limit $q \to 1$.

It is easy to verify that (19) is a solution for the set (17), (18), but the reader may wonder how the solution (19) was obtained and if it is unique. The way we solved (17), (18) is as follows. For a chosen N-value, (17) and (18) are written down for all vectors $|n_0 n_1 n_2\rangle$ with $n_0 + n_1 + n_2 = N$. This gives rise to a large system of nonlinear equations in a number of ordinary variables. For example, when $N = 1$ there remain three equations in two unknowns $\alpha(100)$ and $\beta(001)$, but as N increases the system grows rapidly. For every N, one can try to solve this system. It turns out that for some small values of N the solution is not always unique, but as N increases a unique solution emerges, leading to (19).

This technique can be applied to $\mathrm{gl}_q(l+1)$ with $l > 2$. Explicitly, we looked at symmetric representations $\{N, 0, 0, 0\}$ with basis states $|n_0 n_1 n_2 n_3\rangle$ of $\mathrm{gl}_q(4)$, with E and F given by (6) in terms of three unknown functions μ_1, μ_2 and μ_3 (rather than two unknown functions α and β in the case of $\mathrm{gl}_q(3)$). The condition $[E, F] = [H]$ gives rise to equations similar to (17) and (18). When trying to solve these, a solution was obtained for $N = 1$ and $N = 2$. However, for $N = 3$, leading to a system of 16 nonlinear equations in 16 variables, we are able to show that this system is inconsistent unless $q = 1$ (this involved the help of MACSYMA). So $\mathrm{gl}_q(4)$ does not contain a principal subalgebra, even when all relations are restricted to symmetric representations only. For $\mathrm{gl}_q(l+1)$ with $l > 3$ we have not performed any explicit calculations, but the $\mathrm{gl}_q(4)$ case seems to indicate that none of these algebras contains a principal subalgebra for the symmetric representations.

We continue here with the realization given above. In order to emphasize that we are dealing with a q-generalization of angular momentum, the states $|n_0 n_1 n_2\rangle$ shall now be denoted by $|n_{+1} n_0 n_{-1}\rangle$, the index referring to angular momentum projection.

For such vectors belonging to the totally symmetric representations $\{N, 0, 0\}$ of $\mathrm{gl}_q(3)$ or $\mathrm{u}_q(3)$, there exists a realization in terms of q-boson operators (Macfarlane 1989, Biedenharn 1989). Thus we assume there are three number operators N_{+1}, N_0, N_{-1} and three independent q-boson operators b_i and b_i^+ $(i = +1, 0, -1)$ satisfying

$$[N_i, b_i^+] = b_i^+ \qquad [N_i, b_i] = -b_i \qquad b_i b_i^+ - q^{-1} b_i^+ b_i = q^{N_i}. \tag{22}$$

The basis states are then of the form

$$|n_{+1} n_0 n_{-1}\rangle = \frac{(b_{+1}^+)^{n_{+1}} (b_0^+)^{n_0} (b_{-1}^+)^{n_{-1}}}{\sqrt{[n_{+1}]![n_0]![n_{-1}]!}} |0\rangle \tag{23}$$

with $b_i|0\rangle = 0$ and $N_i|n_{+1} n_0 n_{-1}\rangle = n_i|n_{+1} n_0 n_{-1}\rangle$. The principal subalgebra, here denoted by $\mathrm{so}_q(3)$, follows from (21):

$$\begin{aligned}
L_0 &= N_{+1} - N_{-1} \\
L_{+1} &= q^{N_{-1} - \frac{1}{2}N_0} \sqrt{q^{N_{+1}} + q^{-N_{+1}}}\, b_{+1}^+ b_0 + b_0^+ b_{-1} q^{N_{+1} - \frac{1}{2}N_0} \sqrt{q^{N_{-1}} + q^{-N_{-1}}} \\
L_{-1} &= b_0^+ b_{+1} q^{N_{-1} - \frac{1}{2}N_0} \sqrt{q^{N_{+1}} + q^{-N_{+1}}} + q^{N_{+1} - \frac{1}{2}N_0} \sqrt{q^{N_{-1}} + q^{-N_{-1}}}\, b_{-1}^+ b_0
\end{aligned} \tag{24}$$

compared with (21), we have chosen a different factor for the diagonal element, in order to realise the q-relations which are more familiar to physicists:

$$[L_0, L_{\pm 1}] = \pm L_{\pm 1} \qquad [L_{+1}, L_{-1}] = [2L_0]. \tag{25}$$

It is obvious that (24) is the q-generalization of the so(3) subalgebra of u(3).

In the classical case of $u(3) \supset so(3)$, the symmetric representations $\{N, 0, 0\}$ decompose into so(3) representations (L) with $L = N, N-2, \ldots, 1$ or 0. In the q-generalized case, this decomposition is exactly the same. In fact, we have calculated the matrix elements relating the $so_q(3)$ basis to the q-boson basis (23). For this purpose, the following operator can be introduced:

$$s = (b_0^+)^2 q^{N_{+1}+N_{-1}+1} - \sqrt{\frac{[2N_{+1}]}{[N_{+1}]}\frac{[2N_{-1}]}{[N_{-1}]}}\, b_{+1}^+ b_{-1}^+ q^{-N_0-\frac{1}{2}}. \tag{26}$$

It can be verified that s is an $so_q(3)$ scalar, i.e. $[L_i, s] = 0$ $(i = +1, 0, -1)$. Then we have, in terms of the states (23), that:

$$v(N, L, M) = q^{-[(L+M)(L+M-1)]/4}\left\{\frac{[N+L]!![2L+1]}{[N-L]!![N+L+1]!}[L+M]![L-M]!\right\}^{1/2}$$
$$\times \sum_x q^{(2L-1)x/2} s^{(N-L)/2} \frac{|x, L+M-2x, x-M\rangle}{\sqrt{[2x]!![L+M-2x]![2x-2M]!!}} \tag{27}$$

where x runs from $\max(0, M)$ to $\lfloor (L+M)/2 \rfloor$ in steps of one, $L = N, N-2, \ldots, 1$ or 0, and $M = -L, -L+1, \ldots, +L$. As usual, the symbol $[2t]!!$ stands for $[2t][2t-2] \ldots [2]$. The vectors (27) are genuine orthonormal q-generalized angular momentum states:

$$\begin{aligned} L_0 v(N, L, M) &= M v(N, L, M) \\ L_{\pm 1} v(N, L, M) &= \sqrt{[L \mp M][L \pm M + 1]}\, v(N, L, M \pm 1). \end{aligned} \tag{28}$$

The classical analogue of (27) was given by Sharp *et al* (1969) and by Moshinsky *et al* (1975). There, the so(3) states are in an obvious way related to spherical harmonics. In a following paper, we intend to give more details on the derivation of (27) and to relate the states $v(N, L, M)$ to functions which can be seen as q-generalized spherical harmonics.

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